

FUSION IN THE ENTWINED CATEGORY OF YETTER–DRINFELD MODULES OF A RANK-1 NICHOLS ALGEBRA

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ABSTRACT. We rederive a popular nonsemisimple fusion algebra in the braided context, from a Nichols algebra. Together with the decomposition that we find for the product of simple Yetter–Drinfeld modules, this strongly suggests that the relevant Nichols algebra furnishes an equivalence with the triplet W -algebra in the $(p, 1)$ logarithmic models of conformal field theory. For this, the category of Yetter–Drinfeld modules is to be regarded as an *entwined* category (the one with monodromy, but not with braiding).

1. INTRODUCTION

The idea to construct “purely algebraic” counterparts of vertex-operator algebras (conformal field theories) has a relatively long history [1, 2, 3, 4, 5, 6]. In [7, 8, 9, 10, 11, 12, 13, 14], this idea was developed for nonsemisimple—logarithmic—CFT models, which have been intensively studied recently (see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein). In [28], further, a braided and arguably “more fundamental” algebraic counterpart of logarithmic CFT was proposed. It is given by Nichols algebras [29, 30, 31, 32, 33]; the impressive recent progress in their theory (see [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44] and the references therein) is a remarkable “spin-off” of Andruskiewitsch and Schneider’s program of classification of pointed Hopf algebras.

Associating Nichols algebras with CFT models implies that certain CFT-related structures must be reproducible from (some) Nichols algebras. Here, we take the simplest, rank-1 Nichols algebra \mathfrak{B}_p of dimension $p \geq 2$ and, from the category of its Yetter–Drinfeld modules, extract a commutative associative $2p$ -dimensional algebra on the $\mathbf{x}(r)_v$, $1 \leq r \leq p$, $v \in \mathbb{Z}_2$:

$$(1.1) \quad \mathbf{x}(r_1)_{v_1} \mathbf{x}(r_2)_{v_2} = \sum_{\substack{s=|r_1-r_2|+1 \\ \text{step}=2}}^{p-1-|r_1+r_2-p|} \mathbf{x}(s)_{v_1+v_2} + \sum_{\substack{s=2p-r_1-r_2+1 \\ \text{step}=2}}^p \mathbf{p}(s)_{v_1+v_2},$$

with

$$\mathbf{p}(r)_v = \begin{cases} 2\mathbf{x}(r)_v + 2\mathbf{x}(p-r)_{v+1}, & r < p, \\ \mathbf{x}(p)_v, & r = p. \end{cases}$$

This is the FHST fusion algebra [45] (also see [12]), which makes part of what we know from [11] (also see [46]) to be an equivalence of representation categories—of the *triplet*

algebra $W(p)$ in the $(p, 1)$ logarithmic conformal models [47, 48, 49, 50, 45] and of a small quantum sl_2 at the $2p$ th root of unity, proposed in this capacity in [7, 8] and then used and studied, in particular, in [51, 52, 53, 54, 55] (this quantum group had appeared before in [56, 57, 58]).

The reoccurrence of the fusion algebra in the braided approach advocated in [28], together with some other observations, supports the idea that Nichols algebras are *at least as good as* the quantum groups proposed previously [7, 8, 9, 10, 59] for the logarithmic version of the Kazhdan–Lusztig correspondence (the correspondence between vertex-operator algebras and quantum groups).¹

Algebra (1.1) arises here as an algebra in the center of the category of Yetter–Drinfeld \mathfrak{B}_p -modules; the $\mathfrak{x}(r)_v$ are certain images of the simple Yetter–Drinfeld \mathfrak{B}_p -modules $\mathfrak{X}(r)_v$.² More is actually true: from the study of the representation theory of \mathfrak{B}_p , we obtain that the tensor product of simple Yetter–Drinfeld \mathfrak{B}_p -modules decomposes as

$$(1.2) \quad \mathfrak{X}(r_1)_{v_1} \otimes \mathfrak{X}(r_2)_{v_2} = \bigoplus_{\substack{s=|r_1-r_2|+1 \\ \text{step}=2}}^{p-1-|r_1+r_2-p|} \mathfrak{X}(s)_{v_1+v_2} \oplus \bigoplus_{\substack{s=2p-r_1-r_2+1 \\ \text{step}=2}}^p \mathcal{P}[s]_{v_1+v_2},$$

where $\mathcal{P}[p]_v = \mathfrak{X}(p)_v$ and $\mathcal{P}[r]_v$ for $1 \leq r \leq p-1$ is a reducible Yetter–Drinfeld \mathfrak{B}_p -module with the structure of subquotients

$$(1.3) \quad \begin{array}{ccccc} & & \mathfrak{X}(p-r)_{v+1} & & \\ & \swarrow & & \searrow & \\ \mathcal{P}[r]_v = & \mathfrak{X}(r)_v & & & \mathfrak{X}(r)_{v+2}, \\ & \searrow & & \swarrow & \\ & & \mathfrak{X}(p-r)_{v+1} & & \end{array}$$

Decompositions (1.2) were conjectured in [28] and are proved here. The $\mathfrak{X}(r)_v$ and $\mathcal{P}[r]_v$ do not exhaust all the category of Yetter–Drinfeld \mathfrak{B}_p -modules, but make up “the most significant part of it,” and relations (1.2), together with the structure of $\mathcal{P}[r]_v$, already seem to imply that the category of Yetter–Drinfeld \mathfrak{B}_p -modules is equivalent to the $W(p)$ representation category. This requires an important clarification, however.

In the braided category of Yetter–Drinfeld \mathfrak{B}_p -modules, the simple objects are the $\mathfrak{X}(r)_v$ labeled by $1 \leq r \leq p$ and $v \in \mathbb{Z}_4$ (and, accordingly, $v \in \mathbb{Z}_4$ in $\mathcal{P}[r]_v$, and so on). There are twice as many objects as in the category of $W(p)$ representations [45, 46, 11]. But the presumed equivalence is maintained for *entwined* categories [60]—those endowed with only “double braiding” $D_{y,z} = c_{z,y} \circ c_{y,z}$ (the *monodromy* on the $W(p)$ side). The

¹But the actual motivation in [28], which is yet to be tested on more advanced examples, was that Nichols algebras can actually do better than the “old” quantum groups.

²The notation is fully explained below, but here we note that the module comodule structure, e.g., of $\mathfrak{X}(r)_v$ depends only on r , whereas v serves to distinguish isomorphic module comodules that nevertheless have different braiding.

properties of double braiding can be axiomatized without having to resort to the braiding itself [60]. This defines a *twine structure* and, accordingly, an entwined category. Remarkably, it was noted in [60] that

“many significant notions apparently related to c actually depend only on D or [the twist] θ . The S -matrix, and the subcategory of transparent objects, which play an important role in the construction of invariants of 3-manifolds, are defined purely in terms of the double braiding. More surprisingly, the invariants of ribbon links ... do not depend on the actual braiding, but only on D .”

In the entwined category of Yetter–Drinfeld \mathfrak{B}_p -modules, the objects with v and $v + 2$ in their labels are isomorphic, which sets $v \in \mathbb{Z}_2$ and resolves the “representation doubling problem”; everything else on the algebraic side appears to be already “fine-tuned” to ensure the equivalence. (We do not go as far as modular transformations in this paper, but the above quotation suggests that dealing with entwined categories is not an impediment to rederiving the $W(p)$ modular properties at the Nichols algebra level, in a “braided version” of what was done in [7].)

It may also be worth noting that we derive (1.1) and (1.2) independently (of course, from the same structural results on Yetter–Drinfeld \mathfrak{B}_p -modules, but not from one another). In particular, (1.1) is obtained by directly composing the *action* of $\mathbf{x}(r_1)_{v_1}$ and $\mathbf{x}(r_2)_{v_2}$ on Yetter–Drinfeld modules, with $\mathbf{x}(r)_v : \mathcal{Y} \rightarrow \mathcal{Y}$ given by “running $\mathcal{X}(r)_v$ along the loop” in the diagram (with the notation to be detailed in what follows)



As such, the $\mathbf{x}(r)_v$ depend only on $v \in \mathbb{Z}_2$ —there is no “ \mathbb{Z}_4 option” for them.³

This paper is organized as follows. For the convenience of the reader, we summarize the relevant points from [28] in Sec. 2; a very brief summary is that for a Nichols

³Diagram (1.4) involves not only the squared braiding B^2 of Yetter–Drinfeld modules but also, “in the loop,” the braiding itself (and the ribbon map θ). This does not affect the statement of the equivalence of entwined categories, but rather suggests exploring a further possibility, elaborating on the fact that the braiding of a Yetter–Drinfeld \mathfrak{B}_p -module *with itself* and *with its dual* also depends on $v \in \mathbb{Z}_2$, not $v \in \mathbb{Z}_4$ (and the same for the ribbon map). An entwined’ category might allow these braidings in addition to twines. This is similar to the idea of *twist equivalence* in the theory of Nichols algebras [32] (the similarity is not necessarily superficial if we recall that the braiding of “bare vertex operators” is diagonal for \mathfrak{B}_p).

algebra $\mathfrak{B}(X)$, a category of its Yetter–Drinfeld modules can be constructed using another braided vector space Y (whose elements are here called “vertices,” and the Yetter–Drinfeld modules the “multivertex” modules). In Sec. 3, we introduce duality and the related assumptions that make it possible to write diagrams (1.4). In Sec. 4, everything is specialized to a rank-1 Nichols algebra \mathfrak{B}_p (depending on an integer $p \geq 2$). First and foremost, “everything” includes multivertex Yetter–Drinfeld modules. We actually construct important classes of these modules quite explicitly (Appendix B), which allows proving (1.2) and also establishing duality relations among the modules. We also study their braiding, find the ribbon structure, and finally use all this to derive (1.1) from (1.4) for \mathfrak{B}_p . Basic properties of Yetter–Drinfeld modules over a braided Hopf algebra are recalled in Appendix A.

2. THE NICHOLS ALGEBRA OF SCREENINGS

We summarize the relevant points of [28] in this section.

Screenings and $\mathfrak{B}(X)$. The underlying idea is that the nonlocalities associated with screening operators—multiple-integration contours, such as

$$(2.1) \quad \text{---} \times \text{---} \times \text{---} \times \text{---} \times \text{---} \times \text{---} = \iiint_{-\infty < z_1 < z_2 < z_3 < \infty} s_{i_1}(z_1) s_{i_2}(z_2) s_{i_3}(z_3),$$

where $s_j(z)$ are the “screening currents”—allow introducing a *coproduct* by contour cutting, called “deconcatenation” in what follows:

$$(2.2) \quad \Delta: \text{---} \times \text{---} \times \text{---} \times \text{---} \mapsto \text{---} \times \text{---} \times \text{---} \times \text{---} + \text{---} \times \text{---} \times \text{---} \times \text{---} + \text{---} \times \text{---} \times \text{---} \times \text{---} \\ + \text{---} \times \text{---} \times \text{---} \times \text{---} + \text{---} \times \text{---} \times \text{---} \times \text{---} + \text{---} \times \text{---} \times \text{---} \times \text{---}$$

(with the line cutting symbol subsequently understood as \otimes). A *product* of “lines populated with crosses” is also defined, as the “quantum” shuffle product [61], which involves a *braiding* between any two screenings. It is well known that these three structures—coproduct, product, and braiding—satisfy the braided bialgebra axioms [61]. The *antipode* is in addition given by contour reversal. The braided Hopf algebra axioms are then satisfied for quite a general braiding (by far more general than may be needed in CFT); it is rather amusing to see how the braided Hopf algebra axioms are satisfied by merging and cutting contour integrals [28]. The algebra *generated by* single crosses—individual screenings—is the Nichols algebra $\mathfrak{B}(X)$ of the braided vector space X spanned by the different screening species (whose number is called the rank of the Nichols algebra).

Nichols algebras. The Nichols algebras—“bialgebras of type one” in [29]—are a crucial element in a classification program of *ordinary* Hopf algebras of a certain type (see [30, 32, 31, 37] and the references therein). Nichols algebras have several definitions, whose

equivalence is due to [62] and [30]. The Nichols algebra $\mathfrak{B}(X)$ of a braided linear space X can be characterized as a graded braided Hopf algebra $\mathfrak{B}(X) = \bigoplus_{n \geq 0} \mathfrak{B}(X)^{(n)}$ such that $\mathfrak{B}(X)^{(1)} = X$ and this last space coincides with the space of *all primitive elements* $P(X) = \{x \in \mathfrak{B}(X) \mid \Delta x = x \otimes 1 + 1 \otimes x\}$ and it *generates all of* $\mathfrak{B}(X)$ as an algebra.⁴ Nichols algebras occurred independently in [66], in constructing a quantum differential calculus, as “fully braided generalizations” of symmetric algebras,

$$\mathfrak{B}(X) = k \oplus X \oplus \bigoplus_{r \geq 2} X^{\otimes r} / \ker \mathfrak{S}_r,$$

where \mathfrak{S}_r is the total braided symmetrizer (“braided factorial”).

The space of vertices Y . In addition to the braided linear space X spanned by the different screening species, we introduce the space of vertex operators taken at a fixed point,

$$(2.3) \quad Y = \text{Span}(V_\alpha(0)),$$

where α ranges over the different primary fields in a given CFT model. CFT also yields the braiding $\Psi : X \otimes X \rightarrow X \otimes X$ of any two screenings (which is always applied to two screenings on the same line, as in (2.1)), as well as the braiding $\Psi : X \otimes Y \rightarrow Y \otimes X$ and $\Psi : Y \otimes X \rightarrow X \otimes Y$ of a screening and vertex (also on the same line, as in (2.4) below), and eventually the braiding $\Psi : Y \otimes Y \rightarrow Y \otimes Y$ of any two vertices, but a large part of our construction can be formulated without this last.

The two braided vector spaces X and Y are all that we need in this section; the braiding Ψ can be entirely general.

Dressed vertex operators as $\mathfrak{B}(X)$ -modules. We use the space Y to construct $\mathfrak{B}(X)$ -modules. Their elements are sometimes referred to in CFT as “dressed/screened vertex operators,” for example,

$$(2.4) \quad \begin{array}{c} \times \text{---} \times \text{---} \circ \text{---} \times \end{array} = \iint_{-\infty < x_1 < x_2 < 0} s_{i_1}(x_1) s_{i_2}(x_2) V_\alpha(0) \int_{0 < x_3 < \infty} s_{i_3}(x_3).$$

It is understood that the \times and \circ are decorated with the appropriate indices read off from the right-hand side; but it is in fact quite useful to suppress the indices altogether and let \times and \circ respectively denote the entire spaces X and Y , and we assume this in what follows.

Because the integrations can be taken both on the left and on the right of the vertex position, the resulting modules are actually $\mathfrak{B}(X)$ bimodules. The left and right actions of $\mathfrak{B}(X)$ are by pushing the “new” crosses into the different positions using braiding; the left action, for example, can be visualized as

⁴An important technicality, noted in [63, 64], is a distinction between quantum symmetric algebras [65] and Nichols algebras proper; the latter are selected by the condition that the braiding be *rigid*, which in particular guarantees that the duals X^* are objects in the same braided category with the X .

$$\begin{array}{c} \text{---} \times \text{---} \cdot \text{---} \bigcirc \text{---} \times \text{---} = \\ \times \text{---} \bigcirc \text{---} \times \text{---} + \text{---} \bigcirc \text{---} \times \text{---} \times \text{---} + \text{---} \bigcirc \text{---} \times \text{---} \times \text{---} \end{array}$$

where the arrows, somewhat conventionally, represent the braiding Ψ . Once again by deconcatenation, e.g.,

[illegible]

these bimodules are also bicomodules and, in fact, Hopf bimodules over $\mathfrak{B}(X)$ (see [67, 68, 69, 70] for the general definitions).

Braid group diagrams and quantum shuffles. A standard graphical representation for the multiplication in $\mathfrak{B}(X)$ and its action on its modules is in terms of braid group diagrams. For example, the above left action is represented as (to be read from top down)

$$(2.5) \quad \begin{array}{c} \times \otimes \circ \times \\ | \quad | \quad | \\ \times \quad \circ \quad \times \end{array} \rightarrow \begin{array}{c} \times \quad \circ \quad \times \\ | \quad | \quad | \\ \times \quad \circ \quad \times \end{array} + \begin{array}{c} \times \quad \circ \quad \times \\ | \quad | \quad | \\ \diagdown \quad \diagup \quad \diagdown \end{array} + \begin{array}{c} \times \quad \circ \quad \times \\ | \quad | \quad | \\ \diagup \quad \diagdown \quad \diagup \end{array} = (\text{id} + \Psi_1 + \Psi_2 \Psi_1)(X \otimes Y \otimes X),$$

where we use the “leg notation,” in the right-hand side, letting Ψ_i denote the braiding of the i th and $(i + 1)$ th factors in a tensor product (our notation and conventions are the same as in [28]). The braid group algebra element $\mathbb{W}_{1,2} \equiv \text{id} + \Psi_1 + \Psi_2\Psi_1$ occurring here is an example of quantum shuffles. The product in $\mathfrak{B}(X)$ is in fact the shuffle product

$$(2.6) \quad \mathbb{W}_{r,s} : X^{\otimes r} \otimes X^{\otimes s} \rightarrow X^{\otimes(r+s)}$$

on each graded subspace. The antipode restricted to each $X^{\otimes r}$ is up to a sign given by the “half-twist”—the braid group element obtained via the Matsumoto section from the longest element in the symmetric group:

$$(2.7) \quad \mathcal{S}_r = (-1)^r \Psi_1(\Psi_2 \Psi_1)(\Psi_3 \Psi_2 \Psi_1) \dots (\Psi_{r-1} \Psi_{r-2} \dots \Psi_1) : X^{\otimes r} \rightarrow X^{\otimes r}$$

(with the brackets inserted to highlight the structure, and the sign inherited from reversing the integrations); for example,

$$\mathcal{S}_5 = - \text{diagram}$$

The Hopf bimodules alluded to above are (some subspaces in) $\bigoplus_{r,s \geq 0} X^{\otimes r} \otimes Y \otimes X^{\otimes s}$, with the left and right $\mathfrak{B}(X)$ actions on these also expressed in terms of quantum shuffles as

$$\mathbb{W}_{r,s+1+t} : X^{\otimes r} \otimes (X^{\otimes s} \otimes Y \otimes X^{\otimes t}) \rightarrow \bigoplus_{i=0}^r X^{\otimes(s+r-i)} \otimes Y \otimes X^{\otimes(t+i)}$$

and

$$\sqcup_{s+1+t,r} : (X^{\otimes s} \otimes Y \otimes X^{\otimes t}) \otimes X^{\otimes r} \rightarrow \bigoplus_{i=0}^r X^{\otimes(s+r-i)} \otimes Y \otimes X^{\otimes(t+i)}.$$

Hopf-algebra diagrams. The four operations on bi(co)modules of a braided Hopf algebra \mathfrak{B} are standardly expressed as



which are respectively the left module structure $\mathfrak{B} \otimes \mathcal{Z} \rightarrow \mathcal{Z}$, the left comodule $\mathcal{Z} \rightarrow \mathfrak{B} \otimes \mathcal{Z}$, the right module structure $\mathcal{Z} \otimes \mathfrak{B} \rightarrow \mathcal{Z}$, and the right comodule structures $\mathcal{Z} \rightarrow \mathcal{Z} \otimes \mathfrak{B}$. The product and coproduct in the braided Hopf algebra itself are denoted as \Uparrow and \Downarrow . The braiding is still denoted as \times , but in contrast to the braid-group diagrams, each line now represents a copy of \mathfrak{B} or a \mathfrak{B} (co)module.

Adjoint action and Yetter–Drinfeld modules. The left and right actions of a braided Hopf algebra \mathfrak{B} on its Hopf bimodule \mathcal{Z} give rise to the *left adjoint action* $\mathfrak{B} \otimes \mathcal{Z} \rightarrow \mathcal{Z}$:

$$(2.8) \quad \text{Diagram of left adjoint action} = \text{Diagram with a box labeled } S$$

A fundamental fact is that *the space of right coinvariants in a Hopf bimodule is invariant under the left adjoint action*; this actually leads to an equivalence of categories, the category of Hopf bimodules and the category of Yetter–Drinfeld modules [67, 68, 71, 66]. We recall some relevant facts about Yetter–Drinfeld modules in Appendix A. In our case of modules spanned by dressed vertex operators, the right coinvariants—all those y that map as $y \mapsto y \otimes 1$ under the right coaction—are simply the vertex operators dressed by screenings only from the left, i.e., elements of $X^{\otimes r} \otimes Y$, for example, $\text{---} \times \text{---} \times \text{---} \circ \text{---}$. In terms of *braid group* diagrams (with the lines representing the X and Y spaces), an example of the left adjoint action on such spaces is given by

$$(2.9) \quad \text{Diagram with four vertical lines and crossings} \rightarrow \text{Diagram with four vertical lines} + \text{Diagram with two crossings} + \text{Diagram with two crossings} - \text{Diagram with two crossings} - \text{Diagram with two crossings} - \text{Diagram with two crossings}$$

where a single “new” cross arrives to each of the three possible positions in two ways, one with the plus and the other with the minus sign in front (which is something expected of an “adjoint” action). That the cross never stays to the right of \circ is precisely a manifestation of the above invariance statement for the space of right coinvariants. This means that a number of terms that follow when expressing (2.8) in terms of braid group dia-

We define a modified left action \curlyvee of $\mathfrak{B}(X)$ on its Hopf bimodules spanned by dressed vertex operators by allowing the “new” crosses to arrive only to the left of \circ , for example,

(more crosses might be initially placed to the right of the vertex \circ ; the action does not see them). In general, $\begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array}$ is the map

$$\begin{array}{c} \times \times \times \circ \times \\ | \quad | \quad | \quad | \quad | \end{array} \rightarrow \begin{array}{c} | \quad | \\ | \quad | \end{array} + \begin{array}{c} | \quad | \\ | \quad | \end{array} + \begin{array}{c} | \quad | \\ | \quad | \end{array}$$

(2.13) 

$$(2.14) \quad \triangleright_{r,s} \equiv \sum_{i=0}^r \Psi_{r-i,s+i} (\Psi_{s,i} \Psi_{1,i}^{\uparrow s} \mathcal{S}_i^{\uparrow(s+1)} \Psi_{i,s+1})^{\uparrow(r-i)} : X^{\otimes r} \otimes (X^{\otimes s} \otimes Y) \rightarrow X^{\otimes(s+r)} \otimes Y.$$

(2.15) $\text{---}\times\text{---}\circ\text{---}\times\text{---}\times\text{---}\times\text{---}\circ\text{---}$ or $\text{---}\times\text{---}\circ\text{---}\times\text{---}\times\text{---}\circ\text{---}\times\text{---}\circ\text{---}$

(2.16.2) deconcatenation up to the first \circ .

For multivertex Yetter–Drinfeld modules, the form (2.13) of the adjoint action is valid if \smile is understood as the “cumulative” action preserving right coinvariants; for example,

Fusion product. The multivertex Yetter–Drinfeld modules are not exactly tensor products of single-vertex ones—they carry a different action, which is *not* $(\mu_{\mathcal{Y}} \otimes \mu_{\mathcal{Z}}) \circ \Delta$, and the coaction is not diagonal either. They actually follow via a *fusion product* [28], which is defined on two single-vertex Yetter–Drinfeld modules (each of which is the space of right coinvariants in a Hopf bimodule) as

(2.17) 

$$\sum_{j=0}^t \mathbb{W}_{s,j} \Psi_{1,j}^{\uparrow s} : (X^{\otimes s} \otimes Y) \otimes (X^{\otimes t} \otimes Y) \rightarrow X^{\otimes s} \otimes Y \otimes X^{\otimes t} \otimes Y$$

and then in view of the definition of \vdash , the meaning of (2.17) is that $j \geq 0$ crosses from the right factor are detached from their “native” module and sent to mix with the left crosses (the sum over j is taken in accordance with the definition of the coaction).

The construction extends by taking the fusion product of multivertex modules: the coaction in (2.17) is then the one just described, by deconcatenation up to the first vertex, and the \vdash action on a multivertex module is “cumulative,” i.e., each cross acting from the right, e.g., on $\text{---}\times\text{---}\circ\text{---}\times\text{---}\times\text{---}\circ\text{---}$, arrives at each of the *five* possible positions.

3. DUALITY IN THE CATEGORY OF YETTER–DRINFELD MODULES

We now consider duality in a braided category of representations of a braided Hopf algebra \mathfrak{B} . We briefly recall the standard definitions and basic properties, and then assume that duality exists in the setting of the preceding section; this then allows us to construct endomorphisms of the identity functor in Sec. 4.

3.1. For a \mathfrak{B} -module \mathcal{Z} , we let ${}^\vee\mathcal{Z}$ denote the left dual module in the same (rigid) braided category. The duality means that there are coevaluation and evaluation maps

$$\begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array}$$

which are morphisms in the category and satisfy the axioms

$$\begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} = \text{---} \quad \text{and} \quad \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array} = \text{---}$$

where the two straight lines are $\text{id}_{{}^\vee\mathcal{Z}}$ and $\text{id}_{\mathcal{Z}}$. It follows that

$$\begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} = \begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array}$$

and similarly for the coevaluation.

The dual ${}^\vee\mathcal{Z}$ to a left–left Yetter–Drinfeld \mathfrak{B} -module \mathcal{Z} is a left–left Yetter–Drinfeld \mathfrak{B} -module with the action and coaction, *temporarily* denoted by \curvearrowright and \rightharpoonup , defined as [68]

$$(3.1) \quad \begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} = \begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array} = \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array}$$

The definitions are equivalent to the properties (which, inter alia, imply that the evaluation is a \mathfrak{B} module comodule morphism)

$$(3.2) \quad \begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} = \begin{array}{c} \text{---}\mathcal{Z} \text{---} \\ \text{---}{}^\vee\mathcal{Z} \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array} = \begin{array}{c} {}^\vee\mathcal{Z} \text{---} \\ \mathcal{Z} \text{---} \end{array}$$

We prove the Yetter–Drinfeld property for $\check{\phi}$ and ϕ for completeness. In view of (3.2), it is easiest to verify the Yetter–Drinfeld axiom by establishing that

$$(3.3) \quad \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}$$

Pushing the new action and then the coaction “to the other side,” we see that the left-hand side of (3.3), by the above properties, is equal to

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array}$$

In the first diagram, we insert S at the position of the upper checkmark and S^{-1} into the same line, at the lower checkmark, and use the properties of the antipode,

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array}$$

This readily gives the second diagram above, where we further recognize the right-hand side of the Yetter–Drinfeld axiom assumed for the module. After using it (the third diagram), and after another application of the properties of S and S^{-1} , we obtain the fourth diagram, and it is immediate to see that it coincides with the right-hand side of (3.3) also rewritten by pushing $\check{\phi}$ and ϕ “to the other side.”

3.2. Assuming a rigid category. We further assume that the category of n -vertex Yetter–Drinfeld \mathfrak{B} -modules is rigid; this means that the dual modules are modules in the same category—in our case, multivertex Yetter–Drinfeld \mathfrak{B} -modules, and the action and coaction defined in (3.1) are just those in (2.16)—and hence the evaluation map satisfies the

properties

$$(3.4) \quad \begin{array}{c} \text{Diagram 1: A vertical line with a loop on the left and a box labeled } S \text{ on the right.} \\ \text{Diagram 2: A vertical line with a loop on the right and a box labeled } S \text{ on the left.} \end{array} = \text{and} \quad \begin{array}{c} \text{Diagram 3: A vertical line with a loop on the left and a box labeled } S^{-1} \text{ on the right.} \\ \text{Diagram 4: A vertical line with a loop on the right and a box labeled } S^{-1} \text{ on the left.} \end{array}$$

for any pair of Yetter–Drinfeld \mathfrak{B} -modules. Evidently, we then also have

$$(3.5) \quad \begin{array}{c} \text{Diagram 5: A vertical line with a loop on the left and a box labeled } S \text{ on the right.} \\ \text{Diagram 6: A vertical line with a loop on the right and a box labeled } S \text{ on the left.} \end{array} = \text{and} \quad \begin{array}{c} \text{Diagram 7: A vertical line with a loop on the left and a box labeled } S^{-1} \text{ on the right.} \\ \text{Diagram 8: A vertical line with a loop on the right and a box labeled } S^{-1} \text{ on the left.} \end{array}$$

3.3. If the category ${}^{\mathfrak{B}}_{\mathfrak{B}}\mathcal{YD}$ of Yetter–Drinfeld \mathfrak{B} -modules is rigid, then for each $Z \in {}^{\mathfrak{B}}_{\mathfrak{B}}\mathcal{YD}$, there is a morphism $\chi_Z : \mathcal{Y} \rightarrow \mathcal{Y}$ for any $\mathcal{Y} \in {}^{\mathfrak{B}}_{\mathfrak{B}}\mathcal{YD}$, defined as

$$(3.6) \quad \begin{array}{c} \text{Diagram 9: A vertical line with a box labeled } B^2 \text{ and a box labeled } B \text{ below it. A loop on the right is labeled } Z \text{ and } \vee Z. \\ \text{Diagram 10: A vertical line with a box labeled } B^2 \text{ and a box labeled } B \text{ below it. A loop on the right is labeled } Z \text{ and } \vee Z. \end{array} =$$

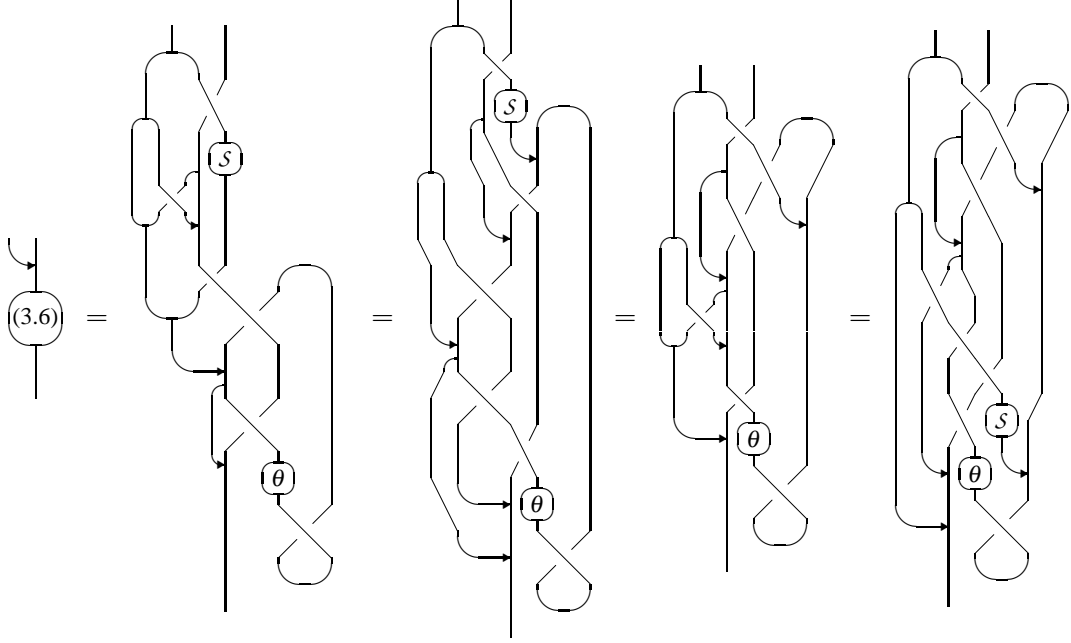
where B is defined in (A.3) and ϑ is any \mathfrak{B} module comodule morphism. In the second diagram, Bespalov’s “squared relative antipode” [67]

$$(3.7) \quad \sigma_2 \equiv \begin{array}{c} \text{Diagram 11: A vertical line with a box labeled } S^2 \text{ and a box labeled } \Pi \text{ below it.} \\ \text{Diagram 12: A vertical line with a box labeled } S^2 \text{ and a box labeled } \Pi \text{ below it.} \\ \text{Diagram 13: A vertical line with a box labeled } S \text{ and a box labeled } \Pi \text{ below it.} \end{array} \quad \text{such that} \quad \begin{array}{c} \text{Diagram 14: A vertical line with a box labeled } S^2 \text{ and a box labeled } \Pi \text{ below it.} \\ \text{Diagram 15: A vertical line with a box labeled } S^2 \text{ and a box labeled } \Pi \text{ below it.} \end{array}$$

(see [68, 73] for its further properties and use) occurs in view of (3.4).

That the map defined by (3.6) is a \mathfrak{B} module comodule morphism follows from the general argument that so are B , evaluation, and coevaluation (and θ). It is also instructive

to see this by diagram manipulation (temporarily writing θ for $\begin{array}{c} \vartheta \\ \otimes \end{array}$ for brevity):



In the first equality, we use only the Yetter–Drinfeld axiom, with B^2 represented by the *first* diagram for B^2 in (A.3); the associativity of action was used in the second equality above; another use of the associativity in the lower part of the third diagram allows recognizing the left-hand side of (A.1); the Yetter–Drinfeld property is then applied in the third equality together with the first property in (3.5), yielding the fourth diagram; there, we use that the property of σ_2 in (3.7) and the first property in (3.4) to obtain the last, fifth diagram, where an “antipode bubble” is annihilated, showing that, indeed,

$$\begin{array}{c} \text{curved arrow} \\ \text{circle (3.6)} \end{array} = \begin{array}{c} \text{circle (3.6)} \\ \text{curved arrow} \end{array}$$

The commutativity of (3.6) with coaction can be verified similarly.

3.4. Ribbon structure. A ribbon structure is a morphism $\vartheta : \mathcal{Y} \rightarrow \mathcal{Y}$ for every object \mathcal{Y} such that

$$(3.8) \quad \begin{array}{c} \vartheta \quad \vartheta \\ \text{circle } B^2 \end{array} = \begin{array}{c} \vartheta \end{array}$$

Whenever it exists, choosing $\vartheta = \vartheta$ in (3.6) makes $\chi_{\mathcal{Z}}$ “multiplicative” in \mathcal{Z} . To show this, we calculate $\chi_{\mathcal{W}}(\chi_{\mathcal{Z}}(\mathcal{Y}))$ by sliding one of the diagrams along the \mathcal{Y} line into the

middle of the other and then expanding:

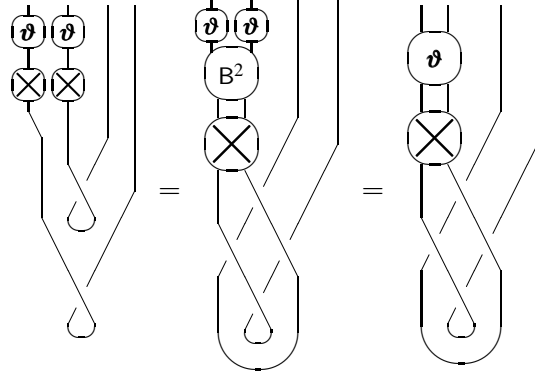
(3.9)

In the last diagram, we recognize the diagonal coaction (the two \checkmark) and action (two \blacktriangleright just below the respective checkmarks) on a tensor product of two Yetter–Drinfeld modules, as

in (A.2). In the bottom right part of the diagram, we recall that $\begin{array}{c} \theta \\ \vdots \\ \theta \end{array} = \begin{array}{c} \theta \\ \times \\ \theta \end{array}$ and calculate

where the first three equalities are elementary (and well-known) rearrangements, the fourth involves (3.4), and the checked equality is verified by repeatedly applying the Yetter–Drinfeld axiom in its right-hand side. The sixth diagram involves B^2 in the upper part and the diagonal action and coaction (A.2) in the lower part, which gives the last

equality. We therefore conclude that if (3.8) holds, then



Substituting this in (3.9) shows that χ is indeed “multiplicative”: $\chi_W(\chi_Z(\mathcal{Y})) = \chi_{W \otimes Z}(\mathcal{Y})$.

4. RANK-ONE NICHOLS ALGEBRA

We specialize the preceding sections to the case of a rank-one Nichols algebra \mathfrak{B}_p , whose relation to the $(p, 1)$ logarithmic CFT models was emphasized in [28]. An integer $p \geq 2$ is fixed throughout.

Notation. We fix the primitive $2p$ th root of unity

$$q = e^{\frac{i\pi}{p}}$$

and introduce the q -binomial coefficients

$$\left\langle \begin{matrix} r \\ s \end{matrix} \right\rangle = \frac{\langle r \rangle!}{\langle s \rangle! \langle r-s \rangle!}, \quad \langle r \rangle! = \langle 1 \rangle \dots \langle r \rangle, \quad \langle r \rangle = \frac{q^{2r} - 1}{q^2 - 1},$$

which are assumed to be specialized to $q = q$.

We sometimes use the notation $(a)_N = a \bmod N \in \{0, 1, \dots, N-1\}$.

4.1. The braided Hopf algebra \mathfrak{B}_p . The rank-1 Nichols algebra \mathfrak{B}_p is $\mathfrak{B}(X)$ for a one-dimensional braided linear space X . We fix an element F (a single screening in the CFT language) as a basis in X . The braiding, taken from CFT, is

$$(4.1) \quad \Psi(F(r) \otimes F(s)) = q^{2rs} F(s) \otimes F(r).$$

Shuffle product (2.6) then becomes

$$F(r) F(s) = \left\langle \begin{matrix} r+s \\ r \end{matrix} \right\rangle F(r+s)$$

and coproduct (2.2) is $\Delta : F(r) \mapsto \sum_{s=0}^r F(s) \otimes F(r-s)$. The antipode defined in (2.7) acts as $S(F(r)) = (-1)^r q^{r(r-1)} F(r)$.

The algebra \mathfrak{B}_p is the linear span of $F(r)$ with $0 \leq r \leq p-1$. It can also be viewed as generated by a single element F , such that $F(r) = \frac{1}{\langle r \rangle!} F^r$, $r \leq p-1$, with $F^p = 0$. We write $F = F(1)$.

Because X is now one-dimensional, we can think of $- \times -$ as just F , and write

$$F(r) = \text{---} \times \text{---} \times \text{---} \times \text{---} \times \text{---} \quad (r \text{ crosses}).$$

4.2. Yetter–Drinfeld \mathfrak{B}_p -modules. We specialize the construction of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules in Sec. 2 to \mathfrak{B}_p . The construction involves another braided vector space Y , a linear span of vertex operators present in the relevant CFT model.

4.2.1. The vertices. For the $(p, 1)$ model corresponding to \mathfrak{B}_p (see [45]), Y is a $2p$ -dimensional space

$$Y = \text{span}(V^a \mid a \in \mathbb{Z}_{4p})$$

with the diagonal braiding

$$(4.2) \quad \Psi(V^a \otimes V^b) = q^{\frac{ab}{2}} V^b \otimes V^a.$$

and with

$$(4.3) \quad \Psi(V^a \otimes F(r)) = q^{-ar} F(r) \otimes V^a, \quad \Psi(F(r) \otimes V^a) = q^{-ar} V^a \otimes F(r).$$

This suffices for calculating the “cumulative adjoint” \mathfrak{B}_p action on multivertex Yetter–Drinfeld modules, as we describe next.

In what follows, the integers a, b, \dots are tacitly considered modulo $4p$.

4.2.2. Multivertex Yetter–Drinfeld \mathfrak{B}_p -modules. We saw in Sec. 2 that multivertex Yetter–Drinfeld modules (see (2.15) and (2.16)) can be represented as an essentially “combinatorial” construction for the crosses to populate, in accordance with the braiding rules, line segments that are separated from one another by vertex operators, e.g., $\text{---} \times \text{---} \circ \text{---} \times \text{---} \times \text{---} \circ \text{---} \times \text{---} \circ \text{---}$, where $\times = X$ and $\circ = Y$ (for a finite-dimensional Nichols algebra, each “segment” can carry only finitely many crosses). In the rank-1 case, each cross can be considered to represent the F element, and each segment is fully described just by the number of the F s sitting there. For example, each two-vertex Yetter–Drinfeld module is a linear span of

$$(4.4) \quad V_{s,t}^{a,b} = \text{---} \times \text{---} \times \text{---} \text{---} \overset{a}{\circ} \text{---} \times \text{---} \overset{b}{\circ} \text{---},$$

where s and t must not exceed $p-1$ ($s=2$ and $t=1$ in the picture) and a and b indicate V^a and V^b . Because the braiding is diagonal, there is a \mathfrak{B}_p module comodule for each fixed a and b (and c, \dots for multivertex modules).

The simplest, one-vertex Yetter–Drinfeld \mathfrak{B}_p -modules are spanned by

$$(4.5) \quad V_s^a = \text{---} \times \text{---} \times \text{---} \times \text{---} \circ \text{---}^a \quad (s \text{ crosses}),$$

where s ranges over a subset of $[0, \dots, p-1]$. The \mathfrak{B}_p coaction is by “deconcatenation up to the first vertex” in all cases, i.e.,

$$\begin{aligned} \delta V_s^a &= \sum_{r=0}^s F(r) \otimes V_{s-r}^a, \\ \delta V_{s,t}^{a,b} &= \sum_{r=0}^s F(r) \otimes V_{s-r,t}^{a,b}, \end{aligned}$$

and similarly for $V_{s,t,u}^{a,b,c}$, and so on.

The \mathfrak{B}_p action (which is the left adjoint action (2.8)) is then calculated as

$$F \blacktriangleright V_s^a = \xi \langle s-a \rangle \langle s+1 \rangle V_{s+1}^a, \quad \xi = 1 - q^2,$$

and the cumulative adjoint evaluates on multivertex spaces as

$$\begin{aligned} F \blacktriangleright V_{s,t}^{a,b} &= \xi \langle s+2t-a-b \rangle \langle s+1 \rangle V_{s+1,t}^{a,b} + \xi q^{2s-a} \langle t-b \rangle \langle t+1 \rangle V_{s,t+1}^{a,b}, \\ F \blacktriangleright V_{s,t,u}^{a,b,c} &= \xi \langle s+2t+2u-a-b-c \rangle \langle s+1 \rangle V_{s+1,t,u}^{a,b,c} \\ &\quad + q^{2s-a} \xi \langle t+2u-b-c \rangle \langle t+1 \rangle V_{s,t+1,u}^{a,b,c} + q^{2s+2t-a-b} \xi \langle u-c \rangle \langle u+1 \rangle V_{s,t,u+1}^{a,b,c}, \end{aligned}$$

and so on.

The braiding follows from (4.1), (4.2), and (4.3), for example,

$$(4.6) \quad \Psi(V_s^a \otimes V_t^b) = q^{\frac{1}{2}(a-2s)(b-2t)} V_t^b \otimes V_s^a.$$

4.3. Module types and decomposition. We now study the category of Yetter–Drinfeld \mathfrak{B}_p -modules in some detail: we find how the one-vertex and two-vertex spaces decompose into indecomposable Yetter–Drinfeld \mathfrak{B}_p -modules. We first forget about braiding and study only *the module comodule structure*; the action and coaction *are* related by the Yetter–Drinfeld axiom, but we try to avoid speaking of Yetter–Drinfeld modules before we come to the braiding.

4.3.1. The relevant module comodules, which we construct explicitly in Appendix **B**, are as follows:

- simple r -dimensional module comodules $\mathcal{X}(r)$, $1 \leq r \leq p$; for $r = p$, we sometimes use the special notation $\mathcal{S}(p) = \mathcal{X}(p)$;

- the p -dimensional extensions

$$(4.7) \quad \mathcal{V}[r] = \begin{array}{ccc} & \mathcal{X}(p-r) & \\ & \swarrow & \\ \mathcal{V}[r] = & \mathcal{X}(r) & \end{array} \quad 1 \leq r \leq p-1,$$

where the arrow means that $\delta\mathcal{X}(p-r) \subset (\text{the “trivial” piece } \mathfrak{B}_p \otimes \mathcal{X}(p-r)) + \mathfrak{B}_p \otimes \mathcal{X}(r)$.

- $2p$ -dimensional indecomposable module comodules $\mathcal{P}[r]$ with the structure of subquotients

$$(4.8) \quad \mathcal{P}[r] = \begin{array}{ccccc} & & \mathcal{X}(p-r) & & \\ & \swarrow & & \searrow & \\ & \mathcal{X}(r) & & \mathcal{X}(r) & \\ & \searrow & & \swarrow & \\ & & \mathcal{X}(p-r) & & \end{array} \quad 1 \leq r \leq p-1.$$

4.3.2. We also show in Appendix **B** that the p^2 -dimensional *one-vertex space*

$$\mathbb{V}_p(1) \equiv \text{Span}(V_s^a \mid 0 \leq a, s \leq p-1)$$

decomposes into \mathfrak{B}_p module comodules as

$$(4.9) \quad \mathbb{V}_p(1) = \mathcal{S}(p) \oplus \bigoplus_{1 \leq r \leq p-1} \mathcal{V}[r]$$

and the p^4 -dimensional *two-vertex space*

$$\mathbb{V}_p(2) \equiv \text{Span}(V_{s,t}^{a,b} \mid 0 \leq a, b, s, t \leq p-1)$$

decomposes as

$$(4.10) \quad \mathbb{V}_p(2) = p^2 \mathcal{S}(p) \oplus \bigoplus_{1 \leq r \leq p-1} 2r(p-r) \mathcal{V}[r] \oplus \bigoplus_{1 \leq r \leq p-1} (p-r)^2 \mathcal{P}[r].$$

Multivertex spaces give rise to “zigzag” Yetter–Drinfeld modules, which we do not consider here.

4.3.3. Notation. Compared with representation theory of Lie algebras, the role of highest-weight vectors is here played by *left coinvariants* V_0^a and $V_{0,t}^{a,b}$. When a module comodule of one of the above types $\mathcal{A} = \mathcal{X}, \mathcal{V}$, or \mathcal{P} is constructed starting with a left coinvariant, we use the notation $\mathcal{A}_0^{\{a\}}$ or $\mathcal{A}_{0,t}^{\{a,b\}}$ to indicate the coinvariant, and sometimes also use the notation such as $\mathcal{X}_{0,t}^{\{a,b\}}(r)$ to indicate the dimension (although it is uniquely defined by a, t, b , and the module type).

4.3.4. The module comodules that can be constructed starting with one-vertex coinvariants V_0^a are classified immediately, as we show in **B.1**. The module comodule *generated*

from V_0^a under the \mathfrak{B}_p action is isomorphic to $\mathcal{X}(r)$ whenever $(a)_p = r - 1$ ($1 \leq r \leq p$). If $r \leq p - 1$, then extension (4.7) follows immediately.

4.3.5. The strategy to classify two-vertex \mathfrak{B}_p module comodules according to their characteristic left coinvariant $V_{0,t}^{a,b}$ is to consider the following cases that can occur under the action of $F(s)$ on the left coinvariant.

- (1) $F(s) \blacktriangleright V_{0,t}^{a,b}$ is nonvanishing and not a coinvariant for all s , $1 \leq s \leq p - 1$. In this case, there are the possibilities that
 - (a) $F(s) \blacktriangleright V_{0,t}^{a,b}$ is a coinvariant, i.e., $F(s) \blacktriangleright V_{0,t}^{a,b} = \text{const } V_{0,t+s}^{a,b}$, for some $s \leq p - 1$, and
 - (b) $F(s) \blacktriangleright V_{0,t}^{a,b}$ is not a coinvariant for any $s \leq p - 1$.
- (2) $F(s) \blacktriangleright V_{0,t}^{a,b} = 0$ for some $s \leq p - 1$. In this case, further possibilities are
 - (a) For some $s' < s$, $F(s') \blacktriangleright V_{0,t}^{a,b}$ is a coinvariant, and
 - (b) $F(s') \blacktriangleright V_{0,t}^{a,b}$ is not a coinvariant for any $s' < s$. We then distinguish the cases where
 - (i) $V_{0,t}^{a,b}$ is in the image of F , and
 - (ii) $V_{0,t}^{a,b}$ is not in the image of F .

We show in Appendix **B** that these cases are resolved as follows in terms of the parameters a , t , and $r = (a + b - 2t)_p + 1$:

- 1a: $1 \leq r \leq p - 1$ and either $t \leq (a)_p - r$ or $(a)_p + 1 \leq t \leq p - r - 1$. Then the left coinvariant is the leftmost coinvariant in (1.3), and the Yetter–Drinfeld module generated from it is the “left–bottom half” $\mathcal{L}(r)$ of $\mathcal{P}[r]$ (see **B.2.3**).
- 1b: $r = p$. Then $\mathcal{X}(p) \equiv \mathcal{S}(p)$ is generated from the left coinvariant.
- 2a: is not realized.
- 2(b)i: $1 \leq r \leq p - 1$ and either $t \geq p - r + (a)_p + 1$ or $p - r \leq t \leq (a)_p$. Then the bottom Yetter–Drinfeld submodule $\mathcal{B}(r)$ in $\mathcal{P}[p - r]$ is generated from the left coinvariant.
- 2(b)ii: $1 \leq r \leq p - 1$ and either $(t \leq (a)_p \text{ and } (a)_p - r + 1 \leq t \leq p - 1 - r)$ or $(t \geq (a)_p + 1 \text{ and } p - r \leq t \leq p - r + (a)_p)$. Then $\mathcal{X}(r)$ is generated from the left coinvariant.

4.4. Braiding sectors. The $\mathcal{X}(r)$ and the other module comodules appearing above satisfy the Yetter–Drinfeld axiom. Considering them as Yetter–Drinfeld \mathfrak{B}_p -modules means that isomorphic module comodules may be distinguished by the braiding. This is indeed the case: for example, shifting $a \rightarrow a + p$ in $\mathcal{X}_0^{\{a\}}$ or $\mathcal{X}_{0,t}^{\{a,b\}}$ does not affect the module comodule structure described in Appendix **B**, but changes the braiding with elements of \mathfrak{B}_p by a sign in accordance with (4.3). We thus have *pairs* $(\mathcal{A}_v, \mathcal{A}_{v+1})$, $v \in \mathbb{Z}_2$, of isomorphic

module comodules distinguished by a sign occurring in their braiding. In particular, there are $2p$ nonisomorphic simple Yetter–Drinfeld modules.

Further, these Yetter–Drinfeld modules can be viewed as elements of a braided category, whose braiding (see (A.3)) involves (4.6). The dependence on a in (4.6) is modulo $4p$, and hence we have not pairs but quadruples $(\mathcal{A}_v)_{v \in \mathbb{Z}_4}$, with the different \mathcal{A}_v distinguished by their braiding with other such modules. In particular, there are $4p$ nonisomorphic simple objects in this braided category of Yetter–Drinfeld \mathfrak{B}_p -modules [28].

It is convenient to write $a = (a)_p - vp$, $v \in \mathbb{Z}_4$ [28], and introduce the notation $\mathcal{X}(r)_v$ for simple modules, with

$$\mathcal{X}_0^{\{a\}} \cong \mathcal{X}(r)_v \quad \text{whenever} \quad a = r - 1 - vp.$$

As before, r is the dimension, and we sometimes refer to v as the braiding sector or braiding index. For $v \in \mathbb{Z}_4$, the isomorphisms are in the braided category of Yetter–Drinfeld \mathfrak{B}_p -modules. The “quadruple structure” occurs totally similarly for other modules, including those realized in multivertex spaces; for example, for any $a, b \in \mathbb{Z}$, we have the isomorphisms among the simple Yetter–Drinfeld modules realized in the two-vertex space (cf. (B.10)!):

$$\mathcal{X}_{0,t}^{\{a,b\}} \cong \mathcal{X}(r)_v \quad \text{whenever} \quad a + b - 2t = r - 1 - vp \quad \text{and} \quad (\text{B.8}) \vee (\text{B.9}) \quad \text{holds.}$$

For the reducible extensions as in (4.7), the two subquotients have adjacent braiding indices, and we conventionally use one of them in the notation for the reducible module:

$$(4.11) \quad \mathcal{V}[r]_v = \begin{array}{c} \mathcal{X}(p-r)_{v+1} \\ \swarrow \\ \mathcal{X}(r)_v \end{array}$$

and $\mathcal{V}_{0,t}^{\{a,b\}}[r]_v \cong \mathcal{V}[r]_v$ whenever $a + b - 2t = r - 1 - vp$ and (B.8) \vee (B.9) holds.

In (4.8), the relevant braiding indices range an interval of three values, and we use the leftmost value in the notation for the entire reducible Yetter–Drinfeld module, which yields (1.3), with $\mathcal{P}_{0,t}^{\{a,b\}}[r]_v \cong \mathcal{P}[r]_v$ whenever $a + b - 2t = r - 1 - vp$ and (B.19) holds.

In the above formulas and diagrams, $v \in \mathbb{Z}_4$ if the modules are viewed as objects of the braided category of Yetter–Drinfeld \mathfrak{B}_p -modules. But *if the Yetter–Drinfeld \mathfrak{B}_p -modules are considered as an entwined category, then the braiding sectors v and $v + 2$ become indistinguishable, and hence $v \in \mathbb{Z}_2$* . In particular, there are $2p$ nonisomorphic simple objects in the entwined category of Yetter–Drinfeld \mathfrak{B}_p -modules.

4.5. Proof of decomposition (1.2). Decomposition (1.2) can be derived from the list in 4.3.5 as follows. The fusion product (2.17) of two one-vertex modules is the map

(assuming that $a, b \leq p-1$ to avoid writing $(a)_p$ and $(b)_p$)

$$(4.12) \quad V_s^a \otimes V_t^b \mapsto \sum_{i=0}^b q^{-ai} \begin{Bmatrix} s+i \\ s \end{Bmatrix} V_{s+i, t-i}^{a,b}.$$

In evaluating $\mathcal{X}_0^{\{a\}}(s) \otimes \mathcal{X}_0^{\{b\}}(t)$, this formula is applied for $0 \leq s \leq a$ and $0 \leq t \leq b$. Then the left coinvariants produced in the right-hand side are $V_{0,u}^{a,b}$, where $0 \leq u \leq b$ and $u \leq a$. But the conditions defining the different items in the list in **4.3.5** have the remarkable property that the module $\mathcal{A}_{0,u}^{\{a,b\}}$ generated from each such coinvariant is as follows:

$$(4.13) \quad \mathcal{A}_{0,u}^{\{a,b\}} = \begin{cases} \mathcal{X}_{0,u}^{\{a,b\}}, & a+b \leq p-1, \\ \mathcal{X}_{0,u}^{\{a,b\}}, & a+b \geq p \text{ and } u \geq a+b-p+2, \\ \mathcal{L}_{0,u}^{\{a,b\}}, & a+b \geq p \text{ and } a+b-2u-p \geq 0, \\ \mathcal{S}_{0,u}^{\{a,b\}}, & a+b \geq p \text{ and } a+b-2u-p = -1, \\ \mathcal{B}_{0,u}^{\{a,b\}}, & a+b \geq p \text{ and } a+b-2u-p \leq -2. \end{cases}$$

This is established (*only* for $0 \leq u \leq a, b \leq p-1$) by direct inspection of each case in the list at the end of **4.3.5**. The module $\mathcal{L}_{0,u}^{\{a,b\}}$ is the “left–bottom half” of $\mathcal{P}_{0,u}^{\{a,b\}}$, and $\mathcal{B}_{0,u}^{\{a,b\}}$ is the bottom sub(co)module in another \mathcal{P} module; the details are given in **B.2.3**. The crucial point is that each $\mathcal{L}_{0,u}^{\{a,b\}}$ can be extended to $\mathcal{P}_{0,u}^{\{a,b\}}$ (while the \mathcal{B} , on the other hand, are not interesting in that they are sub(co)modules in the \mathcal{L} that are already present). We next claim that each of the \mathcal{L} s occurring in $\mathcal{X}_0^{\{a\}}(s) \otimes \mathcal{X}_0^{\{b\}}(t)$ indeed occurs there together with the entire \mathcal{P} module; this follows from counting the dimensions and from the fact that there are no more left coinvariants among the $V_{v,w}^{a,b}$ appearing in the right-hand side of (4.12) (and, of course, from the structure of the modules described in Appendix **B**).

Once it is established that each \mathcal{L} occurs in (4.13) as a sub(co)module of the corresponding \mathcal{P} , it is immediate to see that (4.13) is equivalent to (1.2).

4.6. Duality. We now recall Sec. 3. The structures postulated there are indeed realized for the n -vertex Yetter–Drinfeld \mathfrak{B}_p -modules.

4.6.1. One-vertex modules: coev and ev maps. For the irreducible Yetter–Drinfeld module $\mathcal{X}_0^{\{a\}} \cong \mathcal{X}(r)$ as in **B.1.1**, the coevaluation map $\text{coev} : k \rightarrow \mathcal{X}_0^{\{a\}} \otimes {}^\vee \mathcal{X}_0^{\{a\}}$ is given in terms of dual bases as

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \text{---} \text{---} \text{---} = \sum_{s=0}^{r-1} V_s^a \otimes U_s^{-a}, \quad r = (a)_p + 1,$$

and the evaluation map $\text{ev} : {}^\vee \mathcal{X}_0^{\{a\}} \otimes \mathcal{X}_0^{\{a\}} \rightarrow k$, accordingly, as

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \text{---} \text{---} \text{---} : U_s^{-a} \otimes V_t^b \mapsto 1 \delta_{s,t} \delta_{a,b}.$$

We then use (3.1) to find the \mathfrak{B}_p module comodule structure on the U_s^a . Simple calculation shows that

$$F(r)U_s^a = \mathfrak{q}^{r(r-1)-ra-2rs} \left\langle \begin{matrix} s \\ r \end{matrix} \right\rangle (-\xi)^r \prod_{t=s-r}^{s-1} \langle t+a \rangle U_{s-r}^a,$$

$$\delta U_s^a = \sum_{r=0}^{p-1-s} (-1)^r \mathfrak{q}^{-ra-2sr-r(r-1)} F(r) \otimes U_{s+r}^a.$$

It follows that we can identify $U_s^a = (-1)^{a+s} \mathfrak{q}^{(s+1)(s+a-2)} V_{p-1-s}^{a-2+2p}$ (the action and coaction—and in fact the braiding—are identical for both sides). The coevaluation and evaluation maps can therefore be expressed as

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \sum_{s=0}^{r-1} V_s^a \otimes V_{p-1-s}^{2p-a-2} (-1)^{a+s} \mathfrak{q}^{(s+1)(s-a-2)}, \quad r = (a)_p + 1,$$

and

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} : V_s^a \otimes V_t^b \mapsto \langle V_s^a, V_t^b \rangle = (-1)^s \mathfrak{q}^{-s^2+s(a-1)} \delta_{s+t, p-1} \delta_{a+b, 2p-2}.$$

For $a \not\equiv p-1 \pmod{p}$, evidently, $a = r-1 - vp$ implies that $2p-a-2 = p-r-1 + (v+1)p$, and therefore the module left dual to $\mathcal{V}[r]$ in (4.11), with $r = (a)_p + 1$, can be identified as

$$\begin{array}{c} \cdot \\ \swarrow \quad \searrow \\ \mathcal{V}(\mathcal{V}[r]_v) = \mathcal{X}(r)_{-v} \quad \mathcal{X}(p-r)_{-v-1} \end{array}$$

where $\mathcal{X}(r)_{-v}$ is dual to $\mathcal{X}(r)_v$ in (4.11).

The properties expressed in (3.4) and (3.4) now hold, as is immediate to verify.

4.6.2. Two-vertex modules. Similarly to 4.6.1, for the $U_{s,t}^{a,b}$ that are dual to the two-vertex basis,

$$\langle U_{s,t}^{a,b}, V_{u,v}^{c,d} \rangle = \delta_{a+c,0} \delta_{b+d,0} \delta_{s,u} \delta_{t,v},$$

it follows from (3.1) that

$$F(r)U_{s,t}^{a,b} = \sum_{u=0}^r (-1)^r \mathfrak{q}^{r(r-1)-r(b+2s+2t)} c_{s-r+u, t-u}^{-a, -b}(r, u) U_{s-r+u, t-u}^{a,b}.$$

Replacing here $s \rightarrow p-1-s$ and $t \rightarrow p-1-t$ and noting that the coefficients $c_{s,t}^{a,b}(r, u)$ in (B.4) have the symmetry

$$c_{s,t}^{a,b}(r, u) = \mathfrak{q}^{2r(r+2t+2s-a-b)} c_{p-1-s-r+u, p-1-t-u}^{-a-2, -b-2}(r, u), \quad r \geq u,$$

we arrive at the identification

$$U_{s,t}^{a,b} = (-1)^{t+s} \mathfrak{q}^{(t+s+2)(2a+b+t+s-3)} V_{p-1-s, p-1-t}^{a-2, b-2}.$$

Hence, under the pairing

$$\langle V_{s,t}^{a,b}, V_{u,v}^{c,d} \rangle = (-1)^{s+t} \mathbf{q}^{(s+t)(2a+b+1-s-t)} \delta_{a+c,-2} \delta_{b+d,-2} \delta_{s+u,p-1} \delta_{t+v,p-1},$$

the module left dual to $\mathcal{P}_{0,t}^{\{a,b\}}$ can be identified with $\mathcal{P}_{0,p-r-t-1}^{\{-a-2,-b-2\}}$ (as before, $a+b-2t = r-1-vp$, $1 \leq r \leq p-1$). The module dual to (1.3) has the structure

$$\vee (\mathcal{P}_{0,t}^{\{a,b\}}[r]_v) = \mathcal{P}_{0,p-r-t-1}^{\{-a-2,-b-2\}}[r]_{-2-v} = \begin{array}{ccc} & \mathcal{X}(p-r)_{-v-1} & \\ \swarrow & & \searrow \\ \mathcal{X}(r)_{-2-v} & & \mathcal{X}(r)_{-v} \\ \searrow & & \swarrow \\ & \mathcal{X}(p-r)_{-v-1} & \end{array}$$

4.7. Ribbon structure. We set

$$\boldsymbol{\vartheta} V_s^a = \mathbf{q}^{\frac{1}{2}((a+1)^2-1)} V_s^a,$$

which obviously commutes with the \mathfrak{B}_p action and coaction, and

$$(4.14) \quad \boldsymbol{\vartheta} V_{s,t}^{a,b} = \mathbf{q}^{\frac{1}{2}((a+b-2t+1)^2-1)} \sum_{i=0}^s \mathbf{q}^{-ia} \xi^i \begin{Bmatrix} t+i \\ i \end{Bmatrix} \prod_{j=0}^{i-1} \langle t+j-b \rangle V_{s-i,t+i}^{a,b}$$

(we recall that $\xi = 1 - \mathbf{q}^2$).

4.8. Algebra (1.1) from (1.4). With the above ribbon structure, we now calculate diagram (3.6) in some cases. To maintain association with the diagram, we write $\chi_{\mathcal{Z}}(\mathcal{Y})$ as $\mathcal{Y} \leftarrow \mathcal{Z}$ (the reasons for choosing the right action are purely notational/graphical). The calculations in what follows are based on a formula for the double braiding: for two one-vertex modules, the last diagram in (A.3) evaluates as

$$(4.15) \quad \mathbf{B}^2 \left(V_s^a \otimes V_t^b \right) = \sum_{n=0}^{s+t} \sum_{i=n}^{s+t} \sum_{j=0}^{\min(i,t)} \mathbf{q}^{ab+2j(j-1)+(i-n-1)(i-n)-2bj+a(n-2i-t)} \\ \times \xi^{i-j} \begin{Bmatrix} i \\ j \end{Bmatrix} \begin{Bmatrix} s+t-j \\ s \end{Bmatrix} \begin{Bmatrix} s+t-n \\ i-n \end{Bmatrix} \prod_{\ell=0}^{i-j-1} \langle \ell+j-b \rangle V_{s+t-n,n}^{a,b}.$$

4.8.1. If \mathcal{Y} is irreducible, $\mathcal{Y} \cong \mathcal{X}(r)_v$, then $\mathcal{X}(r)_v \leftarrow \mathcal{Z}$ can only amount to multiplication by a number; indeed, we find that

$$\text{for all } x \in \mathcal{X}_0^{\{a\}} \text{ with } (a)_p \neq p-1, \quad x \leftarrow \mathcal{X}_0^{\{b\}} = \lambda(a,b)x,$$

where

$$\lambda(a,b) = \frac{\mathbf{q}^{(a+1)(b+1)} - \mathbf{q}^{-(a+1)(b+1)}}{\mathbf{q}^{a+1} - \mathbf{q}^{-a-1}}.$$

It is instructive to reexpress this eigenvalue by indicating the representation labels rather than the relevant coinvariants: for $a = r' - 1 - vp$ and $b = r - 1 - vp$, we find

that $\mathcal{X}(r')_{v'} \leftarrow \mathcal{X}(r)_v$ amounts to multiplication by

$$\begin{aligned} \lambda(r', v'; r, v) &= (-1)^{v'(r+1)+vr'+pvv'} \frac{q^{r'r} - q^{-r'r}}{q^{r'} - q^{-r'}} \\ &= (-1)^{v'(r+1)+vr'+pvv'} \sum_{i=1}^r q^{r'(r+1-2i)}. \end{aligned}$$

The last form is also applicable in the case where $r' = p$, and $\mathcal{S}(p)_{v'} \leftarrow \mathcal{X}(r)_v$ amounts to multiplication by

$$\lambda(p, v'; r, v) = (-1)^{(v'+1)(r-1-vp)} r.$$

For $\mathcal{Y} = \mathcal{V}[r]_v$ in (4.11), it may be worth noting that the identity $\lambda(r', v'; r, v) = \lambda(p - r', v' + 1; r, v)$, $1 \leq r' \leq p - 1$, explicitly shows that the action is the same on both subquotients.

4.8.2. Next, the action $\mathcal{P}[r']_{v'} \leftarrow \mathcal{X}(r)_v$ has a diagonal piece, given again by multiplication by $\lambda(r', v'; r, v)$, and a nondiagonal piece, mapping the top subquotient in

$$\begin{array}{ccc} & \mathcal{X}(p - r')_{v'+1} & \\ \swarrow & & \searrow \\ \mathcal{P}[r']_{v'} = \mathcal{X}(r')_{v'} & & \mathcal{X}(r')_{v'+2} \\ & \searrow & \swarrow \\ & \mathcal{X}(p - r')_{v'+1} & \end{array}$$

into the bottom subquotient. Specifically, in terms of the “top” and “bottom” elements defined in (B.16) and (B.17), we have

$$u_t^{a,b}(1) \leftarrow \mathcal{X}(r) = \lambda(r', v'; r, v) u_t^{a,b}(1) + \mu(r', v'; r, v) v_t^{a,b}(r+1),$$

where

$$\begin{aligned} \mu(r', v'; r, v) &= (-1)^{1+v'r+vr'+pvv'} \frac{q - q^{-1}}{(q^{r'} - q^{-r'})^3} \\ &\quad \times \left((q^{r'r} - q^{-r'r})(q^{r'} + q^{-r'}) - r(q^{r'r} + q^{-r'r})(q^{r'} - q^{-r'}) \right). \end{aligned}$$

Because $\leftarrow \mathcal{X}(r)$ commutes with the \mathfrak{B}_p action and coaction, and because $\mathcal{P}_{0,t}^{\{a,b\}}$ is generated by the \mathfrak{B}_p action and coaction from $u_t^{a,b}(1)$, the action of $\mathcal{X}_0^{\{c\}}$ is thus defined on all of $\mathcal{P}_{0,t}^{\{a,b\}}$.

4.8.3. Let $\mathbf{x}(r)_v$ and $\mathbf{p}(r)_v$ be the respective operations $\leftarrow \mathcal{X}(r)_v$ and $\leftarrow \mathcal{P}(r)_v$. We then have relations (1.1), which are the fusion algebra in [45].

We see explicitly from the above formulas that $\mathcal{A}_{v'} \leftarrow \mathcal{X}(r)_v$ depends on both v' and v only modulo 2.

5. CONCLUSION

The construction of multivertex Yetter–Drinfeld $\mathfrak{B}(X)$ -modules has a nice combinatorial flavor: elements of the braided space X populate line intervals separated by “vertex operators”—elements of another braided space Y , as $\text{---}\times\text{---}\circ\text{---}\times\text{---}\times\text{---}\circ\text{---}\times\text{---}\circ\text{---}$. This construction and the $\mathfrak{B}(X)$ action on such objects are “universal” in that they are formulated at the level of the braid group algebra and work for any braiding. However, even for diagonal braiding, extracting information such as fusion from Nichols algebras by direct calculation is problematic, except for rank 1 (and maybe 2). Much greater promise is held by the program of finding the modular group representation and then extracting the fusion from a generalized Verlinde formula like the one in [12]. Importantly, those Nichols algebras that *are* related to CFT (and some certainly are, cf. [72]) presumably carry an $SL(2, \mathbb{Z})$ representation on the center of their Yetter–Drinfeld category.

Going beyond Nichols algebras $\mathfrak{B}(X)$ may also be interesting, and is meaningful from the CFT standpoint: adding the divided powers such as $F(p)$ in our \mathfrak{B}_p case, which are not in $\mathfrak{B}(X)$ but do act on \mathfrak{B}_p -modules, would yield a braided (and, in a sense, “one-sided”) analogue of the infinite-dimensional quantum group that is Kazhdan–Lusztig-dual to logarithmic CFT models viewed as Virasoro-symmetric theories [13, 14].

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APPENDIX A. YETTER–DRINFELD MODULES

In the category of left–left module comodules over a braided Hopf algebra \mathfrak{B} , a Yetter–Drinfeld (also called “crossed”) module [71, 67, 68] is a left module under an action $\curvearrowright : \mathfrak{B} \otimes \mathcal{Y} \rightarrow \mathcal{Y}$ and left comodule under a coaction $\lrcorner : \mathcal{Y} \rightarrow \mathfrak{B} \otimes \mathcal{Y}$ such that the axiom

$$(A.1) \quad \begin{array}{c} \mathfrak{B} \quad \mathcal{Y} \\ \text{[Diagram: A vertical line with a loop on the left labeled } \mathfrak{B} \text{ and a line on the right labeled } \mathcal{Y} \text{, with arrows indicating the action and coaction.]} \end{array} = \begin{array}{c} \mathfrak{B} \quad \mathcal{Y} \\ \text{[Diagram: A vertical line with a loop on the right labeled } \mathfrak{B} \text{ and a line on the left labeled } \mathcal{Y} \text{, with arrows indicating the action and coaction.]} \end{array}$$

holds. The category ${}^{\mathfrak{B}}\mathcal{YD}$ of Yetter–Drinfeld \mathfrak{B} -modules is monoidal and braided. The action and coaction on a tensor product of Yetter–Drinfeld modules are diagonal, respectively given by

$$(A.2) \quad \begin{array}{c} \text{Diagram 1: A vertical line with a loop on the left side, crossing over itself, and then continuing down.} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 2: A vertical line with a loop on the right side, crossing over itself, and then continuing down.} \end{array}$$

For two Yetter–Drinfeld modules, their braiding and its inverse and square are given by

$$(A.3) \quad \begin{array}{c} \text{Diagram 1: A circle labeled B with two vertical lines passing through it.} \end{array} = \begin{array}{c} \text{Diagram 2: Two vertical lines crossing each other, with the left line passing over the right line.} \end{array}, \quad \begin{array}{c} \text{Diagram 3: A circle labeled B^{-1} with two vertical lines passing through it.} \end{array} = \begin{array}{c} \text{Diagram 4: Two vertical lines crossing each other, with the right line passing over the left line.} \end{array}, \quad \text{and} \quad \begin{array}{c} \text{Diagram 5: A circle labeled B^2 with two vertical lines passing through it.} \end{array} = \begin{array}{c} \text{Diagram 6: Two vertical lines crossing each other twice, with the left line passing over the right line in both crossings.} \end{array} = \begin{array}{c} \text{Diagram 7: Two vertical lines crossing each other twice, with the right line passing over the left line in both crossings.} \end{array}$$

APPENDIX B. CONSTRUCTION OF YETTER–DRINFELD \mathfrak{B}_p MODULES

B.1. One-vertex modules. One-vertex Yetter–Drinfeld \mathfrak{B}_p -modules [28] are spanned by V_s^a (see (4.5)) for a fixed $a \in \mathbb{Z}$ with s ranging over a subset of $[0, \dots, p-1]$, under the \mathfrak{B}_p action and coaction given in 4.2.

B.1.1. Simple modules $\mathcal{X}_0^{\{a\}}$. From each left coinvariant V_0^a , the action of \mathfrak{B}_p generates a simple module comodule of dimension $(a)_p + 1$:

$$\mathcal{X}_0^{\{a\}} = \text{Span}(V_s^a \mid 0 \leq s \leq (a)_p)$$

(simply because $F \blacktriangleright V_{(a)_p}^a = 0$ in accordance with the above formulas). The module comodule structure (in particular, the matrix of $F(r) \blacktriangleright$ in the basis of V_s^a , Eq. (B.2)) depends on a only modulo p , and hence there are just p nonisomorphic simple one-vertex module comodules, for which we choose the notation $\mathcal{X}(r)$ indicating the dimension $1 \leq r \leq p$; then there are the \mathfrak{B}_p module comodule isomorphisms

$$\mathcal{X}_0^{\{a\}} \cong \mathcal{X}(r) \quad \text{whenever} \quad (a)_p = r - 1.$$

B.1.2. As noted above, we sometimes use a special notation $\mathcal{S}(p) = \mathcal{X}(p)$.

B.1.3. For each $1 \leq r \leq p-1$, $\mathcal{X}_0^{\{a\}}$ extends to a reducible module comodule $\mathcal{V}[r]$ with $\mathcal{X}_0^{\{a\}} \cong \mathcal{X}(r)$ as a sub(co)module and with the quotient isomorphic to $\mathcal{X}(p-r)$, as shown

in (4.7). In terms of basis, this is

$$(B.1) \quad \begin{array}{c} v^a(r+1) \xrightarrow{F} \dots \xrightarrow{F} v^a(p) \\ \swarrow \\ v^a(1) \xrightarrow{F} \dots \xrightarrow{F} v^a(r) \end{array}$$

where

$$v^a(i) = F^{i-1} \blacktriangleright V_0^a, \quad i \leq r,$$

is a basis in $\mathcal{X}_0^{\{a\}}(r)$, with the last, r th element given by $v^a(r) = C(r)V_{r-1}^a$ with a nonzero $C(r)$, and hence the upper floor starts with the element $v^a(r+1) = C(r)V_r^a$. The downward arrow in (B.1) can be understood to mean $X \mapsto x_1$ whenever $\delta X = \sum_s F(s) \otimes x_s$; this convention is a reasonable alternative to representing the same diagram as

$$\begin{array}{c} \xrightarrow{F(r)} \dots \xrightarrow{F} v^a(r+1) \xrightarrow{F} \dots \xrightarrow{F} v^a(p) \\ \swarrow \quad \searrow \\ v^a(1) \xrightarrow{F} \dots \xrightarrow{F} v^a(r) \end{array}$$

to express the idea that $\delta v^a(r+1) \in \mathfrak{B}_p \otimes \text{Span}(v^a(j) \mid 1 \leq j \leq r+1)$.

The general form of the adjoint action on the one-vertex space is

$$(B.2) \quad F(r) \blacktriangleright V_s^a = \left\langle \begin{matrix} r+s \\ r \end{matrix} \right\rangle \xi^r \prod_{i=s}^{s+r-1} \langle i-a \rangle V_{s+r}^a,$$

B.1.4. We verify that (4.9) holds by counting the total dimension of the modules just constructed:

$$\dim \mathcal{S}(p) + \sum_{r=1}^{p-1} \dim \mathcal{V}[r] = p + (p-1)p = p^2.$$

With the braiding (4.3), each of the above module comodules satisfies the Yetter–Drinfeld axiom.

B.2. Two-vertex modules. A two-vertex Yetter–Drinfeld module is a linear span of some $V_{s,t}^{a,b}$, $0 \leq s, t \leq p-1$, for fixed integers a and b (see (4.4)). The left adjoint action of \mathfrak{B}_p on these is given by

$$(B.3) \quad F(r) \blacktriangleright V_{s,t}^{a,b} = \sum_{u=0}^r c_{s,t}^{a,b}(r,u) V_{s+r-u,t+u}^{a,b},$$

where

$$(B.4) \quad c_{s,t}^{a,b}(r,u) = \xi^r q^{u(2s-a)} \left\langle \begin{matrix} s+r-u \\ r-u \end{matrix} \right\rangle \left\langle \begin{matrix} t+u \\ u \end{matrix} \right\rangle \prod_{i=u}^{r-1} \langle s+i+2t-a-b \rangle \prod_{j=0}^{u-1} \langle t+j-b \rangle.$$

The dependence on b in (B.4) is modulo p , and on a , modulo $2p$. However, $c_{s,t}^{a+p,b}(r,u) = (-1)^u c_{s,t}^{a,b}(r,u)$ and the matrix of $F(r) \blacktriangleright$ in the basis $(V_{s,t}^{a+p,b})_{0 \leq s,t \leq p-1}$ is the same as in

the basis $((-1)^t V_{s,t}^{a,b})_{0 \leq s,t \leq p-1}$; moreover, the coaction is unaffected by this extra sign. Hence, the *module comodule structure depends on both a and b modulo p* .

We arrive at decomposition (4.10) by first listing all module comodules generated from *left coinvariants*

$$V_{0,t}^{a,b} = \text{---} \overset{a}{\circ} \text{---} \times \text{---} \times \text{---} \times \text{---} \overset{b}{\circ} \text{---} \quad (t \text{ crosses}),$$

and then studying their extensions.

In accordance with (B.3), the algebra acts on left coinvariants as

$$F(r) \blacktriangleright V_{0,t}^{a,b} = \sum_{s=0}^r c_t^{a,b}(r,s) V_{r-s,t+s}^{a,b},$$

with the coefficients

$$(B.5) \quad c_t^{a,b}(r,s) = c_{0,t}^{a,b}(r,s) = \xi^r q^{-sa} \begin{Bmatrix} t+s \\ s \end{Bmatrix} \prod_{i=s}^{r-1} \langle i+2t-a-b \rangle \prod_{j=0}^{s-1} \langle t+j-b \rangle.$$

In practical terms, the cases in 4.3.5 can be conveniently studies as follows.

- (1) $F(r) \blacktriangleright V_{0,t}^{a,b}$ is nonvanishing and not a coinvariant for all r , $1 \leq r \leq p-1$.
- (2) $V_{0,t}^{a,b}$ is not in the image of F and $F(r) \blacktriangleright V_{0,t}^{a,b}$ vanishes for some $r \leq p-1$, i.e.,
$$c_t^{a,b}(r,s) = 0, \quad 1 \leq s \leq r.$$
- (3) $F(r) \blacktriangleright V_{0,t}^{a,b}$ is a coinvariant, i.e., $F(r) \blacktriangleright V_{0,t}^{a,b} = \text{const } V_{0,t+r}^{a,b}$, for some $r \leq p-1$, which is equivalent to

$$\begin{cases} c_t^{a,b}(r,s) = 0, & 0 \leq s \leq r-1, \\ c_t^{a,b}(r,r) \neq 0. \end{cases}$$

Let $\beta = (a+b-2t)_p + 1$. For $0 \leq b \leq p-1$, this is equivalent to $b = (2t + \beta - 1 - a)_p$. In fact, every triple (a,b,t) , $0 \leq a,b,t \leq p-1$, can be uniquely represented as

$$(B.6) \quad (a,b,t) = (a, (2t + \beta - 1 - a)_p, t), \quad 1 \leq \beta \leq p.$$

In this parameterization, coefficients (B.5) become

$$c_t^{a,b}(r,s) = \xi^r q^{-sa} \begin{Bmatrix} t+s \\ s \end{Bmatrix} \prod_{i=s+1}^r \langle i-\beta \rangle \prod_{j=1}^s \langle j-\beta+a-t \rangle.$$

and the analysis of the above cases becomes relatively straightforward. The results are as follows.

B.2.1. Irreducible dimension- p modules $\mathcal{S}_{0,t}^{\{a,b\}}(p)$ (case (1)). A simple module comodule of dimension p , isomorphic to $\mathcal{S}(p)$ in B.1.2, is generated under the \mathfrak{B}_p action from a coinvariant $V_{0,t}^{a,b}$ if and only if

$$(B.7) \quad (a+b-2t)_p + 1 = p.$$

When this condition is satisfied, we write $\mathcal{S}_{0,t}^{\{a,b\}}$, or even $\mathcal{S}_{0,t}^{\{a,b\}}(p)$, for this module comodule isomorphic to $\mathcal{S}(p)$.⁵

B.2.2. Reducible dimension- p modules $\mathcal{V}_{0,t}^{\{a,b\}}[r]$ (case (2)). A simple module comodule isomorphic to $\mathcal{X}(r)$ for some $1 \leq r \leq p-1$ is generated under the action of \mathfrak{B}_p from a coinvariant $V_{0,t}^{a,b}$ that is not itself in the image of F if and only if $r = (a+b-2t)_p + 1$ and either of the two conditions holds:⁶

$$(B.8) \quad t \leq (a)_p \quad \text{and} \quad (a)_p - r + 1 \leq t \leq p-1-r,$$

$$(B.9) \quad t \geq (a)_p + 1 \quad \text{and} \quad p-r \leq t \leq p-r+(a)_p.$$

In this case, we write $\mathcal{X}_{0,t}^{\{a,b\}}$ or $\mathcal{X}_{0,t}^{\{a,b\}}(r)$ for the corresponding module comodule:

$$(B.10) \quad \mathcal{X}_{0,t}^{\{a,b\}} \cong \mathcal{X}(r) \quad \text{whenever} \quad r = (a+b-2t)_p + 1 \text{ and (B.8) or (B.9) holds.}$$

Every $\mathcal{X}_{0,t}^{\{a,b\}}$ is further extended as in (4.7), which in terms of basis is now realized as

$$\begin{array}{c} \sum_{s=0}^{r-1} \langle r-1 \rangle! c_t^{a,b}(r-1,s) V_{r-s,t+s}^{a,b} \xrightarrow{F} \dots \\ \swarrow \\ V_{0,t}^{a,b} \xrightarrow{F} \dots \xrightarrow{F} \sum_{s=0}^{r-1} \langle r-1 \rangle! c_t^{a,b}(r-1,s) V_{r-1-s,t+s}^{a,b} \end{array}$$

with the south-west arrow meaning the same as in **B.1.3**; the quotient is isomorphic to $\mathcal{X}(p-r)$. The notation $\mathcal{V}_{0,t}^{\{a,b\}}[r]$ for this dimension- p module comodule explicitly indicates the relevant left coinvariant and the dimension of the sub(co)module; the module comodule structure depends only on r : $\mathcal{V}_{0,t}^{\{a,b\}}[r] \cong \mathcal{V}[r]$.

B.2.3. Three-floor modules $\mathcal{P}_{0,t}^{\{a,b\}}[r]$ (case (3)). We next assume that none of the above conditions (B.7), (B.8), (B.9) is satisfied. An exemplary exercise shows that the negation of $(B.7) \vee (B.8) \vee (B.9)$ is the “or” of the four conditions

$$(B.11) \quad t \geq p-r+(a)_p+1,$$

$$(B.12) \quad p-r \leq t \leq (a)_p,$$

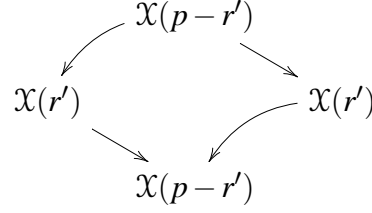
$$(B.13) \quad t \leq (a)_p - r,$$

$$(B.14) \quad (a)_p + 1 \leq t \leq p-r-1,$$

⁵Condition (B.7) is actually worked out as follows: For odd p , it holds if and only if either $t \equiv \frac{1}{2}(a+b+1+p) \pmod{p}$ with $a+b$ even, or $t \equiv \frac{1}{2}(a+b+1)$ with $a+b$ odd. For even p , it holds if and only if either $t \equiv \frac{1}{2}(p+1+a+b) \pmod{p}$ or $t \equiv \frac{1}{2}(a+b+1)$ (which selects only odd $a+b$).

⁶The logic of the presentation is that we assume that $0 \leq a, b \leq p-1$, and hence $(a)_p = a$; but we do not omit the operator of taking the residue modulo p because we refer to formulas given here also in the case where $a \in \mathbb{Z}$.

where again $r = (a + b - 2t)_p + 1$, $1 \leq r \leq p - 1$. The module generated from the coinvariant $V_{0,t}^{a,b}$ is then a sub(co)module in an indecomposable module comodule with the structure of subquotients



where r' is either r or $p - r$, as we now describe.

- i. If $t + r \geq p$ (which means that either (B.11) or (B.12) holds), then the submodule generated from $V_{0,t}^{a,b}$ is isomorphic to $\mathcal{X}(r)$. We let it be denoted by $\mathcal{B}_{0,t}^{\{a,b\}}(r)$. (\mathcal{B} is for “bottom,” and \mathcal{L} is for “left.”)
- ii. If $t + r \leq p - 1$ (which means that either (B.13) or (B.14) holds), then the submodule generated from $V_{0,t}^{a,b}$, denoted by $\mathcal{L}_{0,t}^{\{a,b\}}(r)$, is a p -dimensional reducible module comodule with $\mathcal{B}_{0,t+r}^{\{a,b\}}(p-r) \cong \mathcal{X}(p-r)$ as a sub(co)module and with the quotient isomorphic to $\mathcal{X}(r)$:

$$\mathcal{L}_{0,t}^{\{a,b\}}[r] = \begin{array}{c} \mathcal{X}(r) \\ \searrow \\ \mathcal{X}(p-r). \end{array}$$

In terms of basis, this diagram is

$$V_{0,t}^{a,b} \xrightarrow{F} \dots \xrightarrow{F} \sum_{s=0}^{r-1} \langle r-1 \rangle! c_t^{a,b}(r-1, s) V_{r-1-s, t+s}^{a,b} \xrightarrow{F} \langle r \rangle! V_{0, t+r}^{a,b} \xrightarrow{F} \dots$$

The set of all diagrams of this type actually describes both cases i and ii: according to whether a given coinvariant $V_{0,u}^{a,b}$ is or is not in the image of F , it occurs either in the bottom line (case i) or in the upper line (case ii) of the last diagram.

Every such diagram is extended further, again simply because of the “cofree” nature of the coaction:

$$(B.15) \quad \begin{array}{c} \xrightarrow{\quad} T_{t+r}^{a,b}(r) \\ \searrow \\ V_{0,t}^{a,b} \xrightarrow{F} \dots \xrightarrow{F} \sum_{s=0}^{r-1} \langle r-1 \rangle! c_t^{a,b}(r-1, s) V_{r-s-1, t+s}^{a,b} \xrightarrow{F} \langle r \rangle! c_t^{a,b}(r, r) V_{0, t+r}^{a,b} \xrightarrow{F} \dots \end{array}$$

where, evidently,

$$T_{t+r}^{a,b}(r) = \sum_{s=0}^{r-1} \langle r-1 \rangle! c_t^{a,b}(r-1, s) V_{r-s, t+s}^{a,b}.$$

Setting

$$(B.16) \quad u_t^{a,b}(i) = F^{i-1} \blacktriangleright T_{t+r}^{a,b}(r),$$

$$(B.17) \quad v_t^{a,b}(i) = F^{i-1} \blacktriangleright V_{0,t}^{a,b},$$

we have the full picture extending (B.15) as (omitting the ${}^{a,b}_t$ labels for brevity)

$$\begin{array}{ccccccc} & & u(1) & \xrightarrow{F} \dots \xrightarrow{F} & u(p-r) & & \\ & \swarrow & & & \searrow & & \\ v(1) & \xrightarrow{F} \dots \xrightarrow{F} & v(r) & & & & u(p-r+1) \xrightarrow{F} \dots \xrightarrow{F} u(p) \\ & \searrow & \swarrow & & \swarrow & & \\ & & v(r+1) & \xrightarrow{F} \dots \xrightarrow{F} & v(p) & & \end{array}$$

Here,⁷

$$\delta u(1) = 1 \otimes u(1) + \frac{1}{\langle r-1 \rangle!} F(1) \otimes v(r) + \dots$$

and, similarly,

$$\delta u(p+r-1) = 1 \otimes u(p+r-1) + \frac{q^{-2r}}{\langle r-1 \rangle!} F(1) \otimes v(p) + \dots$$

To label such modules by the leftmost coinvariant $V_{0,t}^{a,b}$ (even though the entire module is not *generated from* this element), we write $\mathcal{P}_{0,t}^{\{a,b\}}$ to indicate both the module type and the characteristic coinvariant. An even more redundant notation is $\mathcal{P}_{0,t}^{\{a,b\}}[r]$, indicating the length r of the left wing (which of course is $r = (a+b-2t)_p + 1$). The module comodule structure depends only on r :

$$(B.18) \quad \mathcal{P}_{0,t}^{\{a,b\}}[r] \cong \mathcal{P}[r].$$

To summarize, given a coinvariant $V_{0,t}^{a,b}$, (B.18) holds if and only if (for $r = (a+b-2t)_p + 1$)

$$(B.19) \quad \begin{aligned} &1 \leq r \leq p-1 \quad \text{and} \\ &t \leq (a)_p - r \quad \text{or} \quad (a)_p + 1 \leq t \leq p-r-1. \end{aligned}$$

B.2.4. Completeness. We verify (4.10) by counting the total dimension of the modules constructed. This gives p^4 , the dimension of $\mathbb{V}_p(2)$, as follows. There are p^2 modules

⁷“The closure of the rhombus” in the above diagram is a good illustration of the use of the Yetter–Drinfeld axiom, which is also used in several other derivations without special notice. The “relative factor” q^{-2r} in the next two formulas, in particular, is an immediate consequence of the Yetter–Drinfeld condition.

$\begin{pmatrix} \mathcal{X}_{0,0}^{\{0,0\}}(1)_0 \\ \mathcal{X}_{0,1}^{\{0,0\}}(4)_1 \\ \mathcal{L}_{0,2}^{\{0,0\}}[2]_1 \\ \mathcal{S}_{0,3}^{\{0,0\}}(5)_2 \\ \mathcal{B}_{0,4}^{\{0,0\}}(3)_2 \end{pmatrix}$	$\begin{pmatrix} \mathcal{X}_{0,0}^{\{0,1\}}(2)_0 \\ \mathcal{S}_{0,1}^{\{0,1\}}(5)_1 \\ \mathcal{X}_{0,2}^{\{0,1\}}(3)_1 \\ \mathcal{L}_{0,3}^{\{0,1\}}[1]_1 \\ \mathcal{B}_{0,4}^{\{0,1\}}(4)_2 \end{pmatrix}$	$\begin{pmatrix} \mathcal{X}_{0,0}^{\{0,2\}}(3)_0 \\ \mathcal{L}_{0,1}^{\{0,2\}}[1]_0 \\ \mathcal{B}_{0,2}^{\{0,2\}}(4)_1 \\ \mathcal{X}_{0,3}^{\{0,2\}}(2)_1 \\ \mathcal{S}_{0,4}^{\{0,2\}}(5)_2 \end{pmatrix}$	$\begin{pmatrix} \mathcal{X}_{0,0}^{\{0,3\}}(4)_0 \\ \mathcal{L}_{0,1}^{\{0,3\}}[2]_0 \\ \mathcal{S}_{0,2}^{\{0,3\}}(5)_1 \\ \mathcal{B}_{0,3}^{\{0,3\}}(3)_1 \\ \mathcal{X}_{0,4}^{\{0,3\}}(1)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{S}_{0,0}^{\{0,4\}}(5)_0 \\ \mathcal{L}_{0,1}^{\{0,4\}}[3]_0 \\ \mathcal{L}_{0,2}^{\{0,4\}}[1]_0 \\ \mathcal{B}_{0,3}^{\{0,4\}}(4)_1 \\ \mathcal{B}_{0,4}^{\{0,4\}}(2)_1 \end{pmatrix}$
$\begin{pmatrix} \mathcal{X}_{0,0}^{\{1,0\}}(2)_0 \\ \mathcal{S}_{0,1}^{\{1,0\}}(5)_1 \\ \mathcal{X}_{0,2}^{\{1,0\}}(3)_1 \\ \mathcal{L}_{0,3}^{\{1,0\}}[1]_1 \\ \mathcal{B}_{0,4}^{\{1,0\}}(4)_2 \end{pmatrix}$	$\begin{pmatrix} \mathcal{X}_{0,0}^{\{1,1\}}(3)_0 \\ \mathcal{X}_{0,1}^{\{1,1\}}(1)_0 \\ \mathcal{X}_{0,2}^{\{1,1\}}(4)_1 \\ \mathcal{X}_{0,3}^{\{1,1\}}(2)_1 \\ \mathcal{S}_{0,4}^{\{1,1\}}(5)_2 \end{pmatrix}$	$\begin{pmatrix} \mathcal{X}_{0,0}^{\{1,2\}}(4)_0 \\ \mathcal{X}_{0,1}^{\{1,2\}}(2)_0 \\ \mathcal{S}_{0,2}^{\{1,2\}}(5)_1 \\ \mathcal{X}_{0,3}^{\{1,2\}}(3)_1 \\ \mathcal{X}_{0,4}^{\{1,2\}}(1)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{S}_{0,0}^{\{1,3\}}(5)_0 \\ \mathcal{X}_{0,1}^{\{1,3\}}(3)_0 \\ \mathcal{L}_{0,2}^{\{1,3\}}[1]_0 \\ \mathcal{B}_{0,3}^{\{1,3\}}(4)_1 \\ \mathcal{X}_{0,4}^{\{1,3\}}(2)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{L}_{0,0}^{\{1,4\}}[1]_{-1} \\ \mathcal{B}_{0,1}^{\{1,4\}}(4)_0 \\ \mathcal{L}_{0,2}^{\{1,4\}}[2]_0 \\ \mathcal{S}_{0,3}^{\{1,4\}}(5)_1 \\ \mathcal{B}_{0,4}^{\{1,4\}}(3)_1 \end{pmatrix}$
$\begin{pmatrix} \mathcal{S}_{0,0}^{\{4,0\}}(5)_0 \\ \mathcal{L}_{0,1}^{\{4,0\}}[3]_0 \\ \mathcal{L}_{0,2}^{\{4,0\}}[1]_0 \\ \mathcal{B}_{0,3}^{\{4,0\}}(4)_1 \\ \mathcal{B}_{0,4}^{\{4,0\}}(2)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{L}_{0,0}^{\{4,1\}}[1]_{-1} \\ \mathcal{B}_{0,1}^{\{4,1\}}(4)_0 \\ \mathcal{L}_{0,2}^{\{4,1\}}[2]_0 \\ \mathcal{S}_{0,3}^{\{4,1\}}(5)_1 \\ \mathcal{B}_{0,4}^{\{4,1\}}(3)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{L}_{0,0}^{\{4,2\}}[2]_{-1} \\ \mathcal{S}_{0,1}^{\{4,2\}}(5)_0 \\ \mathcal{B}_{0,2}^{\{4,2\}}(3)_0 \\ \mathcal{L}_{0,3}^{\{4,2\}}[1]_0 \\ \mathcal{B}_{0,4}^{\{4,2\}}(4)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{L}_{0,0}^{\{4,3\}}[3]_{-1} \\ \mathcal{L}_{0,1}^{\{4,3\}}[1]_{-1} \\ \mathcal{B}_{0,2}^{\{4,3\}}(4)_0 \\ \mathcal{B}_{0,3}^{\{4,3\}}(2)_0 \\ \mathcal{S}_{0,4}^{\{4,3\}}(5)_1 \end{pmatrix}$	$\begin{pmatrix} \mathcal{L}_{0,0}^{\{4,4\}}[4]_{-1} \\ \mathcal{L}_{0,1}^{\{4,4\}}[2]_{-1} \\ \mathcal{S}_{0,2}^{\{4,4\}}(5)_0 \\ \mathcal{B}_{0,3}^{\{4,4\}}(3)_0 \\ \mathcal{B}_{0,4}^{\{4,4\}}(1)_0 \end{pmatrix}$

FIGURE B.1. For each $0 \leq t \leq p-1$, $0 \leq a \leq p-1$, $0 \leq b \leq p-1$ (where $p = 5$), the module comodule generated from $V_{0,t}^{a,b}(r)$ is indicated as $\mathcal{A}_{0,t}^{a,b}(r)_v$, where r is the dimension of the relevant subquotient, v is the braiding index, and \mathcal{A} indicates the module type. Only $a = 0, 1, 4$ are shown for compactness. Whenever an $\mathcal{L}_{0,t}^{\{a,b\}}[r]$ occurs in a column of height 5, the $\mathcal{B}_{0,t+r}^{\{a,b\}}(p-r)$ module is present in the same column. We do not replace negative braiding indices -1 with the “canonical” representative 3 in \mathbb{Z}_4 “for continuity.”

$\mathcal{S}(p)$ constructed in **B.2.1**, $2r(p-r)$ modules $\mathcal{X}(r)$ in **B.2.2** for each $1 \leq r \leq p-1$, making the total of $\frac{1}{3}p(p^2-1)$, and, finally, $(p-r)^2$ modules $\mathcal{L}(r)$ in **B.2.3** for each $1 \leq r \leq p-1$, making the total of $\frac{1}{6}p(p-1)(2p-1)$. Each $\mathcal{S}(p)$ is p -dimensional, each $\mathcal{X}(r)$ extends to a p -dimensional module, and each $\mathcal{L}_{0,t}^{\{a,b\}}[r]$ extends to a $2p$ -dimensional module. The total dimension is

$$p^2 \cdot p + \frac{1}{3}p(p^2-1) \cdot p + \frac{1}{6}p(p-1)(2p-1) \cdot 2p = p^4.$$

B.2.5. Example. Decomposition (4.10) is illustrated in Fig. B.1 for $p = 5$. The figure lists all the modules generated from the $V_{0,t}^{a,b}$ with $a = 0, 1, 4$, $b = 0, 1, 2, 3, 4$, and $t = 0, 1, 2, 3, 4$ (two values of a are omitted for compactness). Each $\mathcal{B}_{0,u}^{\{a,b\}}(r)$ module is a Yetter–Drinfeld submodule in the $\mathcal{L}_{0,u-r}^{\{a,b\}}[p-r]$ module in the same column of $p = 5$ modules. The subscript additionally indicates the braiding sectors (see 4.4).

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